Projective 2D geometry
course 2

Multiple View Geometry

Comp 290-089

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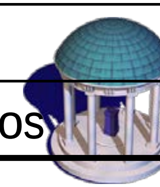
Content

- **Background**: Projective geometry (2D, 3D), Parameter estimation, Algorithm evaluation.
- **Single View**: Camera model, Calibration, Single View Geometry.
- **Two Views**: Epipolar Geometry, 3D reconstruction, Computing F, Computing structure, Plane and homographies.
- **Three Views**: Trifocal Tensor, Computing T.
- **More Views**: N-Linearities, Multiple view reconstruction, Bundle adjustment, auto-calibration, Dynamic SfM, Cheirality, Duality

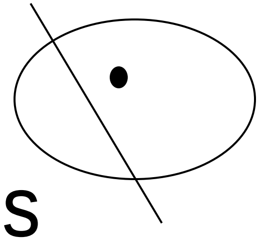
Multiple View Geometry course schedule
(tentative)

Jan. 7, 9	Intro & motivation	Projective 2D Geometry
Jan. 14, 16	(no course)	Projective 2D Geometry
Jan. 21, 23	Projective 3D Geometry	Parameter Estimation
Jan. 28, 30	Parameter Estimation	Algorithm Evaluation
Feb. 4, 6	Camera Models	Camera Calibration
Feb. 11, 13	Single View Geometry	Epipolar Geometry
Feb. 18, 20	3D reconstruction	Fund. Matrix Comp.
Feb. 25, 27	Structure Comp.	Planes & Homographies
Mar. 4, 6	Trifocal Tensor	Three View Reconstruction
Mar. 18, 20	Multiple View Geometry	MultipleView Reconstruction
Mar. 25, 27	Bundle adjustment	Papers
Apr. 1, 3	Auto-Calibration	Papers
Apr. 8, 10	Dynamic SfM	Papers
Apr. 15, 17	Cheirality	Papers
Apr. 22, 24	Duality	Project Demos



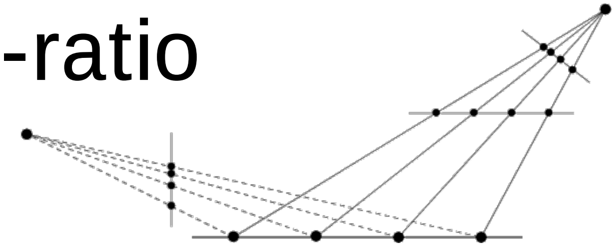
Projective 2D Geometry

- Points, lines & conics



- Transformations & invariants

- 1D projective geometry and the Cross-ratio



Homogeneous coordinates

Homogeneous representation of lines

$$ax+by+c=0 \qquad (a,b,c)^{\mathsf{T}}$$

$$(ka)x+(kb)y+kc=0, \forall k \neq 0 \qquad (a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}$$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3-(0,0,0)^{\mathsf{T}}$ forms $\mathbf{P}^2$

Homogeneous representation of points

$\mathrm{x}=(x,y)^{\mathsf{T}}$ on $\mathrm{l}=(a,b,c)^{\mathsf{T}}$ if and only if $ax+by+c=0$

$$(x,y,1)(a,b,c)^{\mathsf{T}}=(x,y,1)\mathrm{l}=0 \qquad (x,y,1)^{\mathsf{T}} \sim k(x,y,1)^{\mathsf{T}}, \forall k \neq 0$$

The point x lies on the line l if and only if $\mathrm{x}^{\mathsf{T}}\mathrm{l}=\mathrm{l}^{\mathsf{T}}\mathrm{x}=0$

Homogeneous coordinates $(x_1, x_2, x_3)^{\mathsf{T}}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^{\mathsf{T}}$

Points from lines and vice-versa

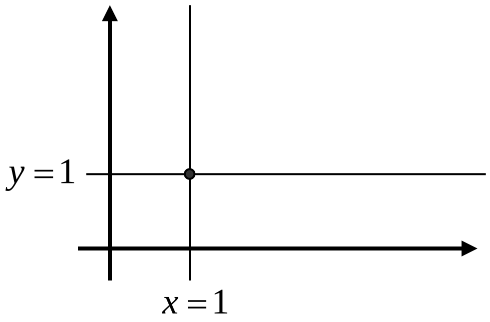
Intersections of lines

The intersection of two lines l and l' is $\mathrm{x} = \mathrm{l} \times \mathrm{l}'$

Line joining two points

The line through two points x and x' is $\mathrm{l} = \mathrm{x} \times \mathrm{x}'$

Example

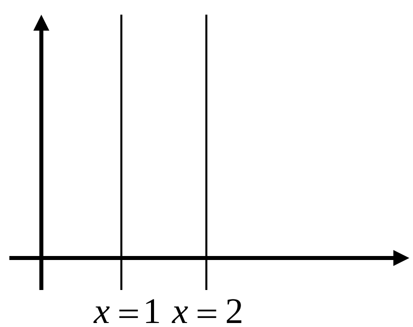


Ideal points and the line at infinity

Intersections of parallel lines

$$\mathrm{l} = (a, b, c)^{\mathsf{T}} \text{ and } \mathrm{l}' = (a, b, c')^{\mathsf{T}} \qquad \mathrm{l} \times \mathrm{l}' = (b, -a, 0)^{\mathsf{T}}$$

Example



$(b, -a)$ tangent vector

(a, b) normal direction

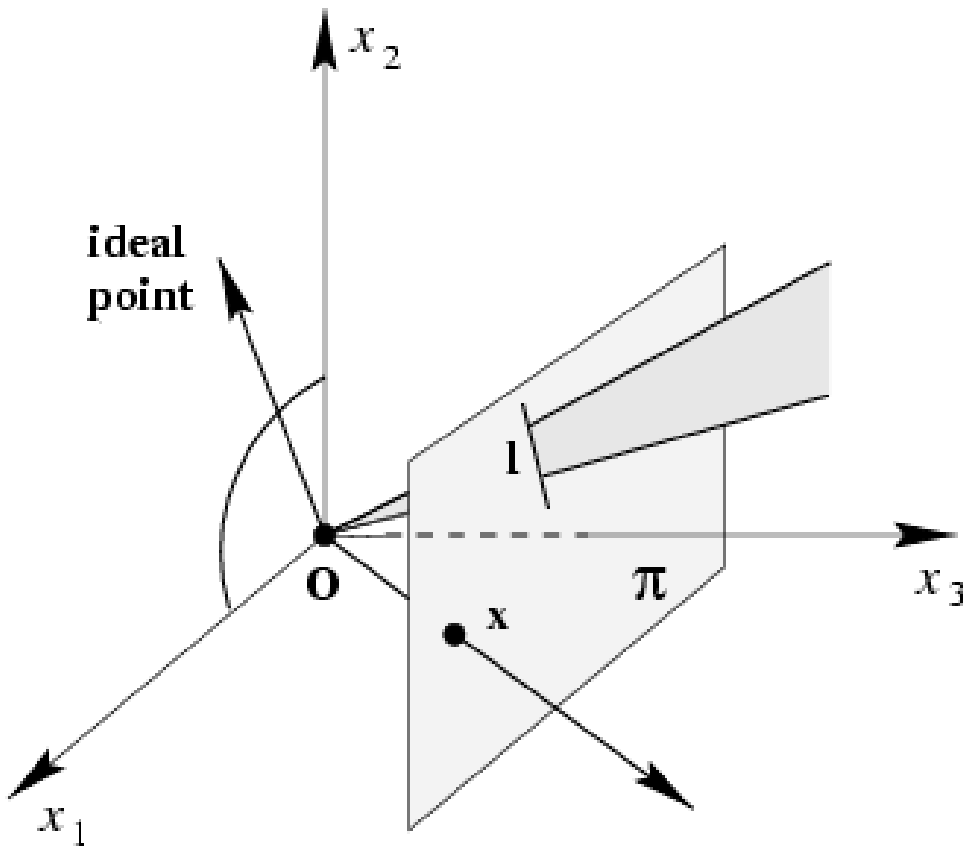
Ideal points $(x_1, x_2, 0)^{\mathsf{T}}$

Line at infinity $\mathrm{l}_\infty = (0, 0, 1)^{\mathsf{T}}$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathrm{l}_\infty$$

Note that in **P**^{2} there is no distinction between ideal points and others

A model for the projective plane



exactly one line through two points

exaclty one point at intersection of two lines

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1{}^2 + bx_1x_2 + cx_2{}^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$a x_i^2 + b x_i y_i + c y_i^2 + d x_i + e y_i + f = 0$$

or

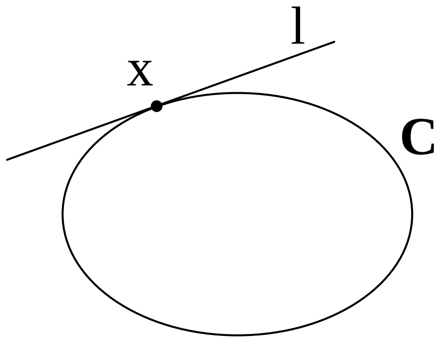
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to $\mathbf{C}$ at point x on $\mathbf{C}$ is given by $\mathrm{l}=\mathbf{C}\mathrm{x}$

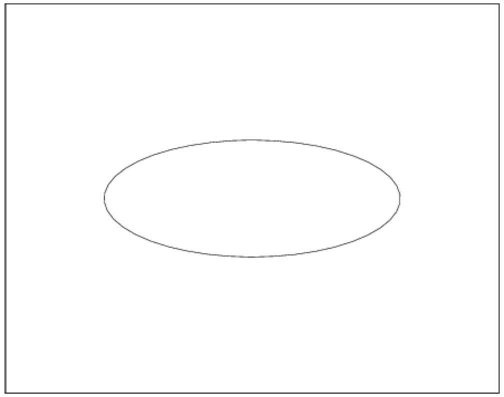


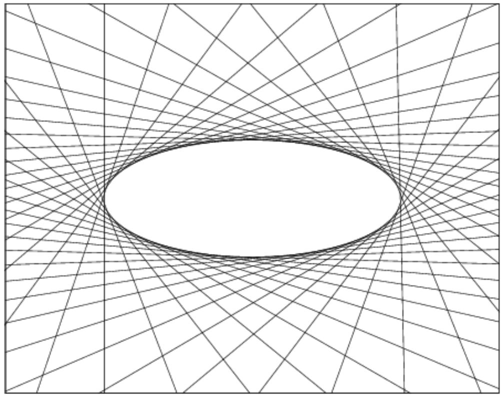
Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*}=\mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes

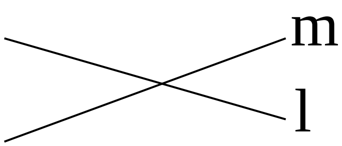




Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

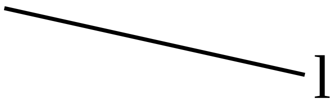
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{lm}^{\mathsf{T}} + \mathrm{ml}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{ll}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Projective transformations

<u>Definition:</u>

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

<u>Theorem:</u>

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 reprented by a vector x it is true that $h(x)=\mathbf{H}x$

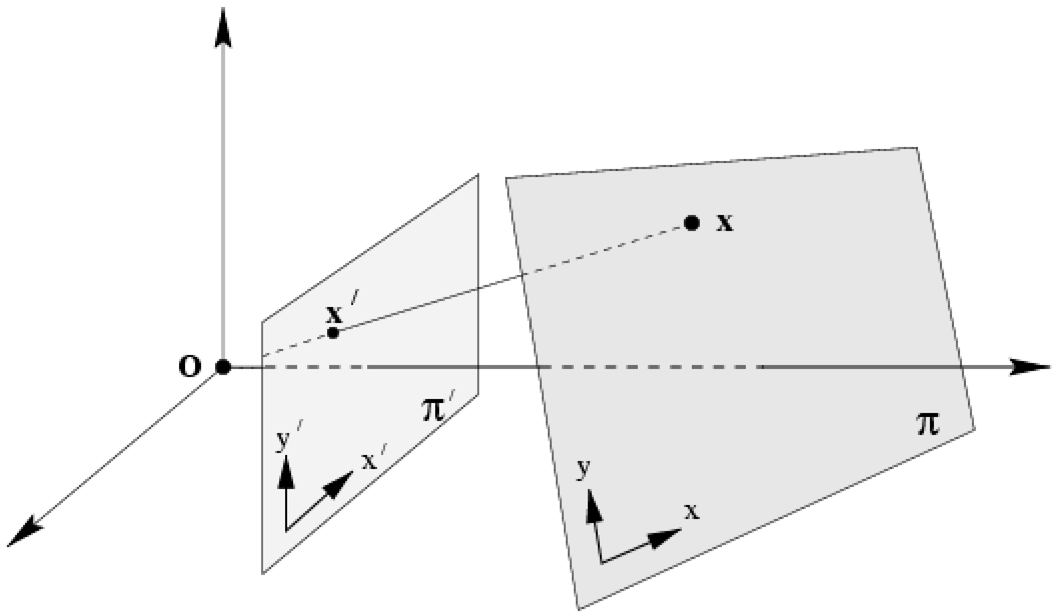
<u>Definition:</u> Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad x' = \mathbf{H}\,x$$

8DOF

projectivity=collineation=projective transformation=homography

Mapping between planes



central projection may be expressed by x’=Hx

(application of theorem)

Removing projective distortion

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

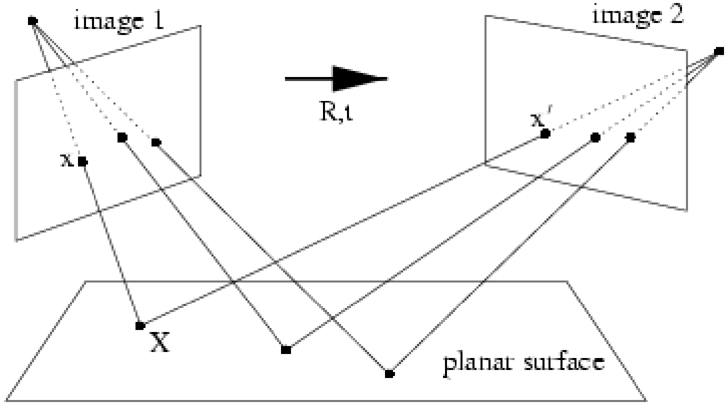
$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

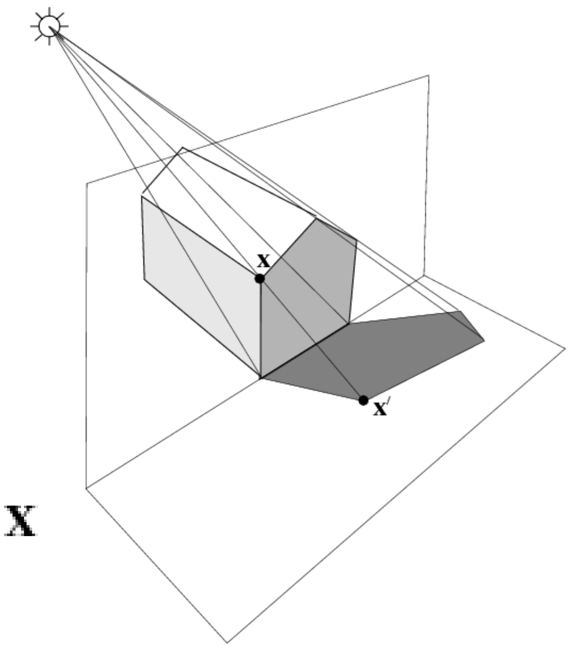
(linear in h_{ij})

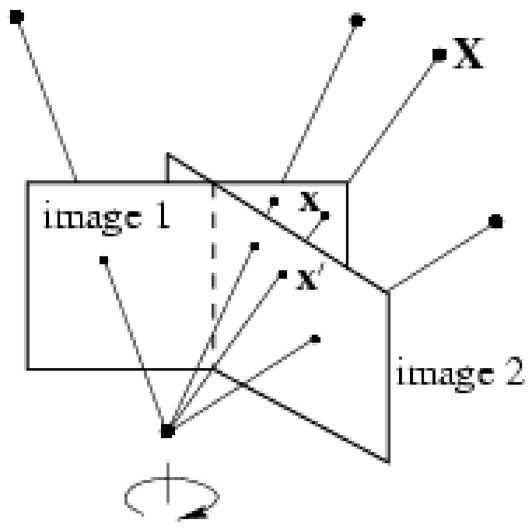
(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary,
better ways to compute (see later)

More examples







Transformation of lines and conics

For a point transformation

$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$$\mathrm{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathrm{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

A hierarchy of transformations

Projective linear group

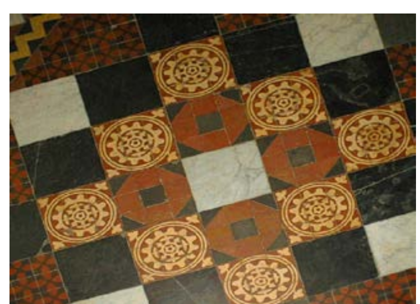
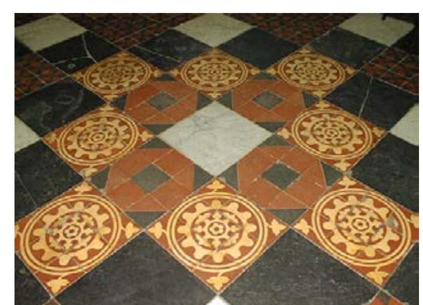
Affine group (last row (0,0,1))

Euclidean group (upper left 2x2 orthogonal)

Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged

Class I: Isometries

(*iso*=same, *metric*=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$
orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

(*isometry + scale*)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

also know as *equi-form* (shape preserving)

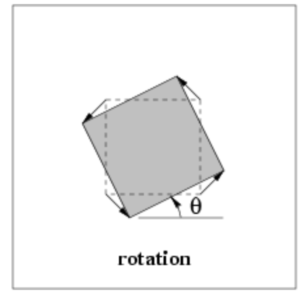
metric structure = structure up to similarity (in literature)

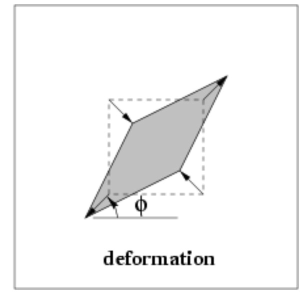
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x}$$





$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths,
ratios of areas

Class VI: Projective transformations

$$\mathrm{x}' = \mathbf{H}_P\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x} \qquad \mathrm{v} = (v_1, v_2)^{\mathsf{T}}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line
(ratio of ratio)

Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,
allows to observe vanishing points, horizon,

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

decomposition unique (if chosen s>0)

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

<u>Example</u>:

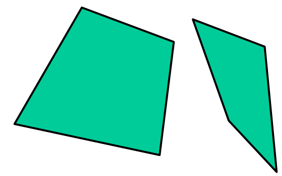
$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Overview transformations

Projective
8dof

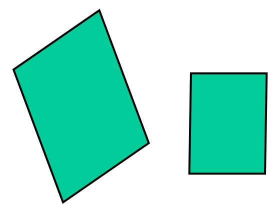
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

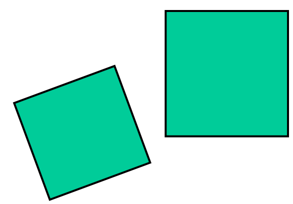
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity l_∞

Similarity
4dof

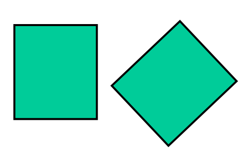
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

Number of invariants?

The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation

e.g. configuration of 4 points in general position has 8 dof (2/pt)

and so 4 similarity, 2 affinity and zero projective invariants